WATER DOWN THE DRAIN
Teacher’s Guide — Getting Started

Purpose
In this two-day lesson, students will collect data from a water dripping experiment. The data that the students collect will be the basis for estimating how much water is wasted from typical leaky faucets. At the beginning of the lesson, the students are faced with a statistic that states leaky faucets in US homes waste $10,000,000 worth of water each year. At the end of the lesson, students will have the opportunity to determine what specifications (homes, faucets, drips/minute) result in that amount of money.

Prerequisites
Students need to understand linear equations, graphing techniques, and unit conversions.

Materials
Required: For each group, water, 2 paper cups, 2 paper clips (one small, one large), a ruler, graduated cylinder, and stopwatch.
Suggested: Graphing paper or a graphing utility.
Optional: Internet access.

Worksheet 1 Guide
The first three pages of the lesson constitute the first day’s work, which consists mainly of data collection. Separate students into groups of four and have them conduct the experiment. Demonstrate proper use of the materials before students begin and emphasize the importance of accurate measurements when gathering data. Each group should record the data in the table provided and should graph the data. Students will produce plots that will lead to a line of best fit for both the “sink” and “tub” faucets and calculate the slopes of these lines.

Worksheet 2 Guide
The third and fourth pages of the lesson constitute the second day’s work in which students will use unit conversions to determine how many are gallons wasted in one day (24 hours) for the sink and tub faucets. The method that the students create will be used to calculate the amount of water wasted in one month (30 days) and one year for both faucets and then applied to national data to determine the amount of water wasted from all households in the country. Questions 11 through 13 are optional since they rely on internet access.

CCSSM Standards
F-LE.1: Distinguish between situations that can be modeled with linear functions and with exponential functions.
F-LE.2: Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).
F-LE.5: Interpret the parameters in a linear or exponential function in terms of a context.
The US Geological Survey estimates that leaky faucets in US homes waste over $10,000,000 worth of water each year! Do you have a leaky faucet in your house? How much water do you think is wasted? How much water do you think a leaky bathroom sink faucet wastes compared to a leaky tub faucet? How would the US Geological Survey reach the conclusion reported?

**Leading Question**
How would you design an experiment to estimate how much water is wasted in US homes?
WATER DOWN THE DRAIN
Student Name: ________________________________ Date: __________________

In your groups, use the materials given to you by your teacher to create a physical model of the situation of a leaking faucet and a leaking bathtub. Each group should have a water source, 2 different sized paper clips, 2 paper cups, a ruler, a stopwatch, and a graduated cylinder.

1. Describe how you initially plan to set up your model. What jobs do each of the materials play? Might you use other materials not provided by your teacher? If so, what are they?

2. Use the model that you have created in your group to fill in the values in the following table:

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<th>Time (in seconds)</th>
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Number of drips during first 10 second interval: ___  Number of drips during first 10 second interval: ___

3. Was your model efficient in its original plan, or did you alter it based on the data you collected?
4. Plot both sets of data and draw a line of best fit for both sets of data.

Title: ______________________________

x-axis: ______________________________

y-axis: ______________________________

5. How would you determine the slope of a line that seems to fit the points best?

Using your method described above:

a) Find the slope of the best fit line for the Sink Faucet data set: _______

b) Find the slope of the best fit line for the Tub Faucet data set: _______
Use the data you collected in the previous day to help answer the following questions.

6. How would you write the equation of each best fit line now that you know the slope?
   Sink Faucet data set equation: ____________

   Tub Faucet data set equation: ____________

7. Using the best fit line equations, describe a method of estimating the amount of water in gallons wasted in one day?

   a) Using your method, how much water does the leaky bathroom sink faucet waste in one day?

   b) How much water does the leaky tub faucet waste in one day?

8. How much water is wasted in one month (30 days) and one year for both faucets?

9. How many households do you think have at least one leaky faucet? The data from Census 2010 (http://www.census.gov/prod/1/pop/p25-1129.pdf) suggests that there are 114.8 million households in the United States. How much water is wasted in one day from all households in the country?
WATER DOWN THE DRAIN

Student Name: ____________________________________________________________ Date: __________________________

10. A family is going on vacation and accidentally left the leaky bathroom sink and tub drains plugged in.

The sink has dimensions:  
Sink depth (in.): 19.125
Sink length (in.): 19.125
Sink width (in.): 8.0

The tub has dimensions:  
Tub depth (in.): 8.625
Tub length (in.): 60
Tub width (in.): 30.25

How long will it take to fill the sink completely? The tub?

11. The US Geological Survey has a drip accumulator calculator that can be found online (http://ga.water.usgs.gov/edu/sc4.html). How do your estimates compare to their calculations? How many drips/minute did you calculate in your experiments?

Using the drip accumulator calculator, how many gallons per day are wasted in
a) 5 Homes, 2 faucets in each, with 60 drips/minute? _______
b) 10,000 homes, 4 faucets in each, with 20 drips/minute? _______

12. On average 1 gallon of tap water costs 1 cent. How much money is wasted per day from the two examples in question 11?

13. What specifications (households, faucets, drips per minute) would give the estimate that US homes waste over $10,000,000 worth of water each year?
The solutions shown represent only some possible solution methods. Please evaluate students’ solution methods on the basis of mathematical validity.

1. Students should be encouraged to try different methods of creating the model given the constraints of each group’s specific materials. One possible method for organizing the groups is to assign roles to each group member as follows:
   - **Student 1:**
     Creates the holes in the paper cups; tests holes for dripping accuracy; counts number of drips during the first 10-second interval.
   - **Student 2:**
     Start stopwatch when the water begins to drip; alerts group at each 10-second interval.
   - **Student 3:**
     Fills cup and covers hole with finger until experiment ready to begin; holds cup over graduated cylinder to measure water lost.
   - **Student 4:**
     Record the amount of water in the graduated cylinder at each 10-second interval.

2. Answers will vary depending on the physical model created by the students but the data should be linear.

3. Efficient models may sometimes be difficult to produce depending on the materials. However, students should think freely about solutions to problems that arise.

4. The tub faucet should have a steeper slope than that of the sink faucet.

5. Finding the average rate of change is an accurate way of determining the slope.

6. Any of the methods of determining the equation of a line work well with this model such as using slope-intercept form. Plotting the points and using a graphing calculator or utility's linear regression can also help to create more mathematically accurate equations.

7. With the x variable representing time (in seconds) in many equations, calculating the number of seconds in a day and then evaluating the functions created in question 6 should give the answer in milliliters. A conversion is necessary to compute the answer in gallons.

8. Multiply the answers in question 7 by the number of days (30) in a month.

9. Estimates on the number of houses with leaky faucets will vary. Determine a reasonable estimate, and then multiply the estimate by the total number of US households, and then by the average amount of water wasted per faucet.

10. The sink has volume of 2,926.125 in³ and the tub has volume 15,654.375 in³. Evaluate your equations created in question 6 for y = these volumes. Conversions may be necessary.

11. Answers will vary depending on the models built. The drip accumulator website will give answers of 57 and 76,089 gallons wasted.

12. 57 cents and $760.89.

13. Answers will vary depending on the number of households/faucets/drips per minute. One solution is 1,000,000 households, with one faucet dripping at 30 drips per minute.
Sinks and tubs are naturally modeled as if they were boxes (that is, rectangular parallelepipeds), but liquids often come in other containers, which can give rise to questions of some interest. (An amusing sidelight – we won’t do any more with it at this point: it is well known that the most economical shape to enclose a given volume is a sphere. So why don’t they make spherical milk bottles? Seriously, what criteria should a packaging method satisfy?)

Your typical plastic or paper cup for a drink may not be in the shape of a cylinder, but more likely a section of a cone. Some small paper cups next to a water fountain go all the way down to the vertex of a cone; most have circular cross sections, which are smaller at the bottom than at the top. We would call the shape a frustum of a cone. Suppose you want such a cup half-full: how high should you fill it? If you fill up to just half the height, you will clearly have it less than half-full, for every cross-sectional circle below the halfway point in height is smaller than every such circle above the halfway point. So if you want the cup half-full, you will have to fill it to more than half the height. How much more?

Let us assume that the cross-sectional radius grows linearly with height. This is a fairly good model even though it ignores the lip at the top for drinking, and some special circles to make gripping the cup easier. A particular brand of cup (Solo) has a diameter of 6.0 cm on the bottom, 9.0 cm at the top, and is 11.8 cm high. Changing to radii rather than diameters because the familiar formulas are in terms of radii leaves us with the bottom radius \( r_0 = 3.0 \) cm and the top radius \( r_1 = 4.5 \) cm. A formula for the radius at height \( h \) (where \( h \) is between 0 and 11.8 cm) that fits these numbers within the accuracy of 0.1 cm is

\[
r = 3.0 + 0.13h.
\]

As we said, it won’t do to set \( r = 3.75 \) cm, which is halfway up. It turns out that what we need is a radius of 3.9 cm, and this comes at a height of 6.9 cm. We want to see a convenient way to find this and then to generalize the result. The formula found in solid geometry texts gives the volume, \( v \), of the frustum of a cone of revolution radius, \( r \), at one end and radius, \( r' \), at the other, with \( a \), the altitude, as

\[
v = \frac{1}{3} \pi a (r^2 + rr' + r'^2).
\]

The typical proof of this requires calculus because it is based on the fact that the volume of a frustum of a cone is the limit of volumes of frustums of inscribed rectangular polygonal pyramids. Those familiar with calculus will recognize the above formula for \( v \) more intuitively as

\[
v = \frac{a}{(r-r_0)} \int_{r_0}^{r} \pi x^2 dx.
\]

We again see the relevance of integral calculus to the formula for the frustum of a cone. Our question was “When is the cup going to be half full?” At height \( h \) along the axis of the cup, the radius of the circular cross-section is \( r(h) = r_0 + mh \), where in our problem \( r_0 = 3 \) cm and \( m = 0.13 \). Then:

\[
r - r_0 = mh \quad \text{and} \quad \frac{h}{r-r_0} = m.
\]

So the volume \( v(h) \) from the bottom up to height \( h \) is given by

\[
v(h) = \pi \int_{0}^{h} (3.0 + 0.13x)^2 dx = \frac{\pi}{0.39} [(3 + 0.13h)^3 - 3].
\]

We want to find \( h \) so that this is half of the volume of the cup, which is

\[
\frac{\pi}{0.39} [(4.5)^3 - 3].
\]
So we get

\[(3 + 0.13h)^3 - 3^3 = \frac{1}{2} [(4.5)^3 - 3^3].\]

Approximating the right-hand side of the equation to 59 leads to \(h \approx 6.9\) cm.

All this can of course be carried out more generally, but there is an interesting wrinkle at the end. We get:

\[(r_0 + mh)^3 = \frac{1}{2} (r_1^3 + r_6^3),\]

and we can find \(h\) by taking cube roots of both sides. But a modeler would also reason as follows: if \(r_0\) equaled \(r_1\), the \(h\) for half a cup would be exactly \((r_0 + r_1)/2\). So it would be natural to want to estimate how far away from \((r_0 + r_1)/2\) the answer is if \(r_1\) does not equal \(r_0\). So let \((r_0 + r_1)/2 = A\) and \((r_1 - r_0)/2 = B\). Then the previous formula becomes

\[(r_0 + mh)^3 = \frac{1}{2} [(A + B)^3 + (A - B)^3] = A^3 + 3AB^2.\]

Remember that we expect \(B\) to be smaller (perhaps much smaller) than \(A\). Then we can argue:

\[r_0 + mh = (A^3 + 3AB^2)^{1/3} = A[1 + 3(B/A)^3]^{1/3} \approx A[1 + (B^2/A)^2],\]

where the right-hand side of the last approximate equality is the first two terms of a binomial expansion with exponent 1/3. So the answer to our "modeler's question", namely "what's a good simple back-of-the-envelope approximation to our answer?" is just \(A + B^2/A\). So as a good approximation the cup will be half-full at about \(B^2/A\) above the midpoint in height; a simple satisfying answer.

In our problem, \(A\) is 3.75 cm and \(B\) is 0.75 cm, so that \(B^2/A = 0.15\) cm. Summing these two we get 3.9 which is just what we got before!