Purpose
In this two-day lesson, students determine their best-matched college. They use decision-making strategies based on their preferences and ranked choices. This lesson guides students through the process of selecting a list of choices and rating these choices based on their preferences in order to find the college most suited to their preferences and requirements.

Prerequisites
Students must understand how information is sorted in matrices or arrays and they should have experience with problem solving in elementary algebra and utilizing open-ended questioning in mathematics.

Materials
*Required*: A current issue of *US World & News Report 100 Best Colleges & Universities* (or similar resource).
*Suggested*: Spreadsheet software (such as MS Excel), internet access.
*Optional*: None.

Worksheet 1 Guide
The first three pages of the lesson constitute the first day’s work in which students generate a list of suitable criteria to help them select a college or university. Students use mathematics to show their preferences of one criterion over another, which may be considered in the model. A set of colleges to consider is determined, and students rate each of the colleges in the set based on how well they meet their preferences for each of the criteria. An initial model for choosing the best school is created.

Worksheet 2 Guide
The fourth and fifth pages of the lesson constitute the second day’s work in which students are introduced to and create a decision matrix using the ratings determined on the first day. This becomes a refined model used for determining the best-suited college. Finally, students are introduced to column vectors and use them to weigh each of the important criteria to create a more refined model. The students are not specifically introduced to multiplication of a matrix by a vector, although they are led toward it.

CCSSM Addressed
N-Q.2: Define appropriate quantities for the purpose of descriptive modeling.
CHOOSING A COLLEGE

Student Name:_______________________________________ Date:_____________________

Making decisions can sometimes be quite difficult, especially when it’s a decision about where you will spend the next two to four years of your life after you graduate from high school – we’re talking about college, of course!

Leading Question
How can you choose the most suitable college for you?
CHOOSING A COLLEGE

Student Name:_____________________________________________ Date:_____________________

1. What criteria are important for you in considering a college or university? Choose 3 – 5 of the criteria that are the most important ones in your opinion.

2. Of the criteria you have chosen, which are more important to you? For instance, is tuition more important than location, or is location more important than tuition? List your preferences in order of importance. Explain why each criterion is more important than the next.

3. Choose 3 – 5 colleges in which you are interested and indicate how well they match or meet each of your chosen criteria. Use reference materials such as the US News & World Report 100 Best Colleges & Universities or similar resource about colleges to help guide you.

Some examples of criteria to consider are athletics, academics, costs, financial aid available, and location.

What makes one criterion more important to you than another?

Think about a rating scheme like GPA in which $A = 4$, $B = 3$, $C = 2$, $D = 1$, and $F = 0$. 

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Think about a rating scheme like GPA in which $A = 4$, $B = 3$, $C = 2$, $D = 1$, and $F = 0$.
4. Use your responses to question 3 to create a model that will help you choose the best school for you.

5. Does your model help you determine which college is best for you? Does it give you your expected results? Does it organize your opinions conveniently? Do you think anyone could use it to help determine their best college choice?

6. If your friend has a different list of schools and preferences that he wants to test, how can you use your model to help him? Be specific.
A **decision matrix** is a tool used to manage a large number of preferences in a simple form. The entries of the decision matrix indicate how well each alternative meets the criterion in question. The rows represent **alternatives** (the objects that are being compared) and the columns represent **criteria** (the characteristics on which the alternatives are being judged). Mathematical operations are used on decision matrices to help reach conclusions about questions related to real-life situations, such as choosing a college.

7. Use your preferences to create a decision matrix for your criteria and the colleges you are considering.

8. Using the decision matrix, how can you determine the final rating of a specific college?

If it looks like College A should be rated higher than College B, what mathematics can you use to show that the individual ratings in each preference “add up” to be more for one than the other?

9. Think of your initial model. If you didn’t use a decision matrix, use one to model a method to determine the best college for you. Do the results make sense? If so, how do they make sense? If not, why do you think they do not make sense? Compare your initial model to this new method. If you used a decision matrix model initially, what led you to do so?
In linear algebra, a column (or row) vector is a matrix consisting only of a single column (or row). Mathematical operations with vectors are used on matrices to help allow for the easy analysis of preference matrices.

10. Look at the relationship between each of your chosen criteria. How can you use mathematics to show that you prefer one criterion over another? Did the decision matrix model you created give equal consideration to all of your criteria? Explain how your model gives either equal or unequal consideration to the criteria and which of these two options should be used in the model.

11. If your model should give unequal consideration to different criteria and it does not, use your responses to question 10 to create a column vector to help you weigh each criteria against one another. The vector, \( \vec{v} \) should indicate how you've given unequal consideration to each of your criteria and the \( i \)th row should correspond to the \( i \)th criterion.

12. Use your decision matrix and column vector to create a modified model and determine the best college for you. What does this model say about the best college for you?

13. Can other real-life decisions be determined using decision matrices? If so, list them and describe briefly how you would go about creating a model for each.
The solutions shown represent only some possible solution methods. Please evaluate students’ solution methods on the basis of mathematical validity.

1. There are various criteria that may be considered. In this answer key we will consider academics, financial aid, and the location of the college as the most important.

2. Here, financial aid is the most important, followed by academics, then location.

3. A scoring or point system can be used in which colleges that meet a given criterion perfectly are given 5 points and colleges that do not meet the criterion at all are given 0 points. So if College I has great financial aid, is a decent school academically, but is a bit far away from home, it will be given the scores 5, 3, and 2 for financial aid, academics, and location, respectively. Similar lines of reasoning gives College II the scores 2, 4, and 5; College III is given the scores 1, 5, and 5; and College IV is given the scores 4, 4, and 0.

4. Each college has been rated on each of the criteria and this information can be summarized in an array, as shown below. The college with the highest sum in its row is the best college choice. Colleges II and III are the best options thus far, but this particular model does not have a method of breaking ties.

<table>
<thead>
<tr>
<th></th>
<th>Financial Aid</th>
<th>Academics</th>
<th>Location</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>College I</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>College II</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>11</td>
</tr>
<tr>
<td>College III</td>
<td>1</td>
<td>5</td>
<td>5</td>
<td>11</td>
</tr>
<tr>
<td>College IV</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>8</td>
</tr>
</tbody>
</table>

5. The model helps determine better schools, but it has flaws because it cannot help break a tie and determine a single best college. Notice it considers each criterion to have equal value. A person who cares about financial aid much more than academics and location, but still considers those to be the three most important criteria, will not necessarily be satisfied with this model.

6. The same model can be used to help, although different criteria will need to be chosen, his set of colleges to consider will be different, and his ratings will be different. He still has no way to break a tie.

7. The decision matrix associated with these rankings is similar to the array given above. It is shown below.

\[
\begin{pmatrix}
5 & 3 & 2 \\
2 & 4 & 5 \\
1 & 5 & 5 \\
4 & 4 & 0
\end{pmatrix}
\]

8. The final rating of a college is the sum of the entries in its row. The college with the highest rating is the best one for the student.

9. The decision matrix model and the initial model provided here are very similar, so they have similar benefits and flaws.

10. One way to indicate preference of one criterion over another is to give different point values to them. If academic quality is considered to be about twice as important as location, and financial aid is considered to be a small bit more important than academics, then we can give each criterion a point value. So we can give ratings of 5 points to financial aid, 4 points to academics, and 2 points to location. The initial model gives equal consideration to all criteria, but it probably should not.
CHOOSING A COLLEGE
Teacher’s Guide — Possible Solutions

11. Most people won’t consider all of their most important criteria all to be equally important, so there should probably be some weighting system in place. The points given in answer 10, namely 5, 4, and 2 points for financial aid, academics, and location, respectively, yield a column matrix whose entries sum to 1. Thus,

\[
\hat{\mathbf{v}} = \begin{pmatrix}
\frac{5}{11} \\
\frac{4}{11} \\
\frac{2}{11}
\end{pmatrix}
\approx \begin{pmatrix}
.45 \\
.36 \\
.18
\end{pmatrix}
\]

12. The decision matrix model can be revised by multiplying the decision matrix by the column vector. This will give weights to each of the entries. The product of the multiplication gives College I a rating of 3.69, College II a rating of 3.24, College III a rating of 3.15, and College IV a rating of 3.24. This model gives College I the highest rating because it meets the financial aid criterion the best and it was comparably rated in the initial model. Thus, College I is the best choice using this model. Note that if a student had chosen to give different ranges of point values for different criteria, then this step would unnecessarily inflate ratings. (So if, for example, financial aid ratings were between 0 and 25, academics ratings were between 0 and 20, and location ratings were between 0 and 10, then this step would give too much weight to some criteria and not enough to others.) Thus, it is important for the student to recognize if their initial model or decision matrix model had already taken the relative importance of each criterion into account.

13. This model can be modified to use to help solve many real-life decision problems. One could choose which political candidate is the right one for them to vote for, for instance. In this case, the alternatives would be each candidate and the criteria would be political issues. The candidates would be rated on how well each of their views or voting records matched with the voter in question. The model could be refined to incorporate the importance of each issue to the voter using matrix multiplication by a column vector in which the entries represent the relative importance of each issue to one another for the voter.
CHOOSING A COLLEGE
Teacher’s Guide — Extending the Model

Our method for choosing a college came in two steps: in the first, we created a decision matrix and rated each of four Colleges (I, II, III, and IV in our example) on each of three criteria (financial aid, academics, and location in our example); in the second, we assigned a weight to each criterion, specifically, 5/11 for financial aid, 4/11 for academics, and 2/11 for location. Notice that these are proportions that add up to 1. In the example, College I was the winner, II and IV were tied for second, and III came in last.

Suppose that instead of (5/11, 4/11, 2/11) as the weights, we had used (x, y, z) with the conditions that each of x, y, and z are greater than or equal to 0, and that the sum of x, y, and z is 1. Can our choice of the four colleges be the winner given that we have appropriately chosen values of x, y, and z? We can’t be sure, but we can find out. Remembering the decision matrix, we see that the score for College I will be 5x + 3y + 2z. Since it’s a lot easier to work with visual representations, and since it’s a lot easier to draw pictures in two dimensions than in three, we can try to reduce our work from three dimensions to two. We know that z = 1 - x - y. If we make that substitution, we get scores, S, for each of the colleges in just two variables: for College I, the score is $S_I = 5x + 3y + 2(1 - x - y) = 3x + y + 2$; for College II, $S_{II} = -3x - y + 5$; for College III, $S_{III} = -4x + 5$; and for College IV, $S_{IV} = 4x + 4y$.

So College I will win if $S_I = \max(S_I, S_{II}, S_{III}, S_{IV})$. If we want to plot our results, the region of the $(x, y)$ plane in which we look is given by the conditions $x \geq 0, y \geq 0$, and also $z = 1 - x - y \leq 0$, which we rewrite as $x + y \leq 1$. Together, these form an isosceles right triangle T in the first quadrant. What will happen is that this triangle will be divided into at most four polygonal regions, and in each of those regions, one of the four colleges will be the winner. In this case, we do get four regions, which means that with the right choice of $(x, y, z)$, any one of the four can be the winner. This will not always happen. Each polygonal region is convex and any segment of each boundary is a segment of a straight line $S_i = S_j$ (where i and j stand for Roman numerals), or else a segment of the three boundary edges of the triangle T. The picture is given on a separate page. A line $S_i = S_j$ divides the plane into two half-planes: in one, $S_i < S_j$, and in the other, $S_i > S_j$. Unless the line happens to go through the origin, a lazy way to tell which is which is to see where the origin should be. (In our example, only the line $S_{II} = S_{III}$ goes through the origin.) This makes each of the four polygonal regions the intersection of half-planes determined by its boundary segments and the boundaries of T.

There is a good chance that this extension of the module could be of tactical value to a student. Suppose she really wants to go to College III, but has pressure from outside sources to choose a different one. Rather than choosing her preference outright, it might be more impressive and help her make her case to say, “Well, I set up my decision matrix, and then I made the choice that financial aid was 2/10 of my personal weight, academics was 5/10, and location was 3/10. When I set $x = 0.2$ and $y = 0.5$, I ended up smack in the middle of the region in which College III was the best! It just happened!”

When you look at the picture, however, you see that it would be much more difficult to end up with College IV as the best choice. There is only a small triangular region in which College IV is preferred to each of the other three, and you would have to pick something very near $x = 0.35$ and $y = 0.6$ (and therefore $z = 0.05$) to end up there.

Note that it is possible to use a different coordinate system so that all three of x, y, and z can be seen at the same time. These are called barycentric coordinates; they are not well known, and it would take a major project to see how they work.
CHOOSING A COLLEGE

Teacher’s Guide — Extending the Model

A graph of each of the combinations of scores set equal to one another so that $S_i = S_j$ and one of the lines defining triangle, $T$.

A graph showing each of the four polygonal regions in which each of the colleges can attain a maximum score compared to the others.