UNSTABLE TABLE
Teacher’s Guide — Getting Started

Purpose
Have you ever tried to eat on an unstable, tippy table? No doubt drinks and soup were spilled easily! Restaurant wait staff often fold paper napkins to wedge under one of the legs to stabilize the table.

In this two-day lesson, students learn to stabilize a table without the use of napkins — they can rotate it up to 90°. The result is counterintuitive but can be verified mathematically.

Prerequisites
Knowledge of slope and continuous functions.

Materials
Required: Small furniture such as doll furniture, construction paper, scissors, and string.
Suggested: None.
Optional: None.

Worksheet 1 Guide
The first four pages of the lesson constitute the first day’s work. Students are encouraged to experiment with small furniture to check to see if they can stabilize it by a rotation in various spots around the classroom. Students develop a model in two dimensions that will help them understand the situation more completely. Students experiment with the model and find the commonalities between the two- and three-dimensional worlds. Finally, they begin to build an intuitive understanding of the Intermediate Value Theorem.

Worksheet 2 Guide
The fifth through eighth pages of the lesson constitute the second day’s work. Students continue to work with the two-dimensional model, but the situation becomes more complicated — it is the two-dimensional version of a 4-legged table in three dimensions. They find through experimentation that it always is possible to stabilize a 3-legged table in two dimensions and give a mathematical explanation that relies on the Intermediate Value Theorem. Finally, they extend their model to the situation at hand (a 4-legged table in three dimensions) and mathematically show that it always is possible to stabilize the 4-legged table.

CCSSM Addressed
A-CED.1: Create equations in one variable and solve them.
F-IF.4: For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of quantities, and sketch graphs showing key features given a verbal description of the relationship.
F-BF.4: Write a function that describes a relationship between two quantities.
F-LE.5: Interpret the parameters of a linear function in terms of context.
Have you ever tried to eat a bowl of soup on an unstable, wobbly table? What happened? If you were in a restaurant, a waiter may have wedged a folded paper napkin under one of the table’s legs to stabilize it — but there’s another way! This is because the problem usually isn’t with the table’s legs; the problem is that the floor is uneven!

**Leading Question**

How can a restaurant’s wait staff use the unevenness of the floor to help them stabilize an unstable table?
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1. It seems that most of the instability in tables is caused by uneven floors. Experiment by placing furniture with 3 or 4 legs in different places around the classroom. Is your furniture unstable? If so, try rotating it little by little. Does it become stable? Repeat this experiment several times in different spots around the classroom. Fill in the table below.

<table>
<thead>
<tr>
<th>Trial #</th>
<th>Degree of Rotation Needed to Stabilize the Table</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
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<tr>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

2. Do you think some rotation will always cause the table to become stable? Why or why not?

3. If you were unable to stabilize the table, it could be that one leg is shorter than the others. Stretch string between the tips each pair of opposite legs. How can you tell if the tips of the legs are coplanar?

What does “coplanar” mean?
What do you know about things that are coplanar?
If the tips of the table are coplanar, it will be stable when the floor is level. If you conduct more trials by rotating a table on an uneven floor, you should observe that rotation always seems to stabilize the table — but to show that it is true requires a mathematical model. Sometimes, to get started, it helps to model a similar but simpler situation.

Two-dimensional objects are usually simpler to study than three-dimensional ones. Even though the two-dimensional tables aren’t useful in the real world, they may be helpful in the mathematical world. In the two-dimensional world, 2- and 3-legged tables would look like the pictures below.

4. What should represent an uneven floor in the two-dimensional world? Use construction paper to cut out an uneven two-dimensional floor and several two-dimensional tables.

What properties might an uneven floor have? Would it keep rising forever or would it rise and fall and stay around the same height?

5. In a two-dimensional model, a rotation in three dimensions must be replaced by a “slide.” Slide a 2-legged two-dimensional table along the two-dimensional floor until both legs contact the floor. Try this for several different starting positions. Is it difficult to stabilize the table? Explain your findings.
6. Do you observe a common property between a 2-legged two-dimensional table and a 3-legged three-dimensional table? Explain your thoughts.

7. What do you observe about changes in the slope of the top of the 2-legged table as it slides along an uneven floor?

8. If the slope of the tabletop is positive at one point and negative at another, what must happen in between? Explain what this tells you about the tabletop.
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9. Consider the 3-legged two-dimensional table on the uneven two-dimensional floor. Slide it until all 3 legs contact the floor and record the length of the slide needed to stabilize the table. Repeat this experiment several times starting at different places on the floor. Record your results.

<table>
<thead>
<tr>
<th>Trial #</th>
<th>Length of Slide Needed to Stabilize the Table</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td></td>
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<td>2</td>
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<td>3</td>
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</tr>
</tbody>
</table>

Consider the length of the slide in terms of the distance between adjacent legs.

10. Was it always possible to stabilize the 3-legged table on the uneven two-dimensional floor? Explain.

11. What do your trials indicate about the length of the slide required?

12. Consider two slope functions: \( l_1 \), the slope of the line from the first leg to the floor at a point below the second leg, and \( l_2 \), the slope of the line from the third leg to the floor below the second leg. An example is shown below. In the example, the slope of \( l_1 \) is negative. Is the slope of \( l_2 \) positive or negative? Explain.

Consider the length of the slide in terms of the distance between adjacent legs.
**UNSTABLE TABLE**

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13. Let $S_1$ and $S_2$ be slope functions that have different values as the table slides along the floor. Subtract these two functions to obtain $S = S_1 - S_2$. Is $S$ a continuous function? Explain.

14. If $S$ is continuous, what must occur between points where $S > 0$ and $S < 0$? Explain.

15. At the point where $S = 0$, what must be true of $S_1$ and $S_2$? What does that tell you about the position of the middle leg with respect to the uneven floor?

16. Suppose the first leg of the table is above the floor while the two other legs touch the floor, as shown below. What slopes would you use to show that as the two-dimensional table slides along the floor, at some position all three legs will touch?

![Diagram of an unstable table with one leg above the floor and two legs touching the floor.](image)
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If you have understood how two-dimensional tables slide along an uneven two-dimensional floor, you should be able to extend the two-dimensional model to three dimensions. Begin by thinking of the legs of a 4-legged three-dimensional table as the table is rotated on the uneven floor. Actually, if a two-dimensional "floor" is bent to form a circle, it’s just like the arc around which a three-dimensional table rotates.

17. Will a 4-legged three-dimensional table always have 3 of its legs touching the uneven floor? Explain.

18. Can a continuous function be found that is positive somewhere and negative somewhere else? If so, what would that tell you about the function?

19. Experiment! Perhaps two or more slope functions will suffice. Since 3 legs of an unstable 4-legged table always will touch the floor, exactly 1 leg always will be above the floor, say, by k mm. Connect the opposite legs of the 4-legged table that do touch the floor with a line segment, l₁. At each end, the height above the floor is 0 mm. To create l₂, connect the third leg with the point on the floor below the fourth leg (the one that doesn't touch the uneven floor). The slope of line l₁ is 0 as is the slope of line l₂ − k, that is, S₁ = 0 and S₂ = −k. Subtract these two functions to obtain S = S₁ − S₂. Of course, the values of S₁, S₂ and S change as the table is rotated. How do the values of S₁ and S₂ change when the table is rotated exactly 90°?
20. Considering what happens to $S_1$ and $S_2$ when the 4-legged table is rotated by exactly 90°, what must happen to $S$ in between? What does this mean in terms of the table?

21. What can you say about the possibility of stabilizing a 4-legged table on an uneven floor? Are you surprised by what your model shows?
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Teacher’s Guide — Possible Solutions

The solutions shown represent only some possible solution methods. Please evaluate students’ solution methods on the basis of mathematical validity.

1. Answers will vary. However, 3-legged tables should always be stable and never need to be rotated because any 3 points define a plane; 4-legged tables should never require more than a 90° rotation.
2. Most students will believe that it is not always possible to stabilize a table with a rotation. Contrary to what students believe, a rotation always will stabilize a table with legs whose ends are coplanar on a surface that is not always increasing or always decreasing.
3. The tips of the legs are coplanar if the strings, when pulled taut, do not bend.
4. Below is a sketch of a possible uneven two-dimensional floor. The floor will rise and fall a bit, but it will generally stay around the same height.

5. There should be no difficulty stabilizing a 2-legged table in two dimensions. It should be stable in any position it is placed.
6. The feet of a 2-legged table always define a line (as any 2 points define a line). The feet of a 3-legged table always define a plane. The concept of a line in two dimensions is similar to the concept of a plane in three dimensions.
7. The slope of the tabletop will change from negative, to 0, to positive, to 0, to negative, and so on as long as it keeps sliding.
8. The slope must be 0 at some point in between. This means that the table eventually will not only be stable, but will also be level.
9. Answers will vary. The length of the slide never should be longer than the distance between adjacent (consecutive) legs.
10. It is always possible to stabilize the table and, in fact, it always can be done with a slide whose length is less than or equal to the distance between adjacent legs.
11. The slide never was longer than the distance between adjacent legs.
12. The slope of \( l_2 \) is positive. Unless they are both 0, the slopes of \( l_1 \) and \( l_2 \) will always have opposite signs.
13. Yes, \( S \) is a continuous function since both \( l_1 \) and \( l_2 \) are continuous functions and subtraction is a continuous operation.
14. Since \( S \) is continuous, it must be 0 at some point in between.
15. If \( S = 0 \), then \( S_1 = S_2 \) and the middle leg must be touching the floor — the table will be stable.
16. The slopes of \( l_1 \) and \( l_2 \) still are used. In the picture, one must find \( l_1 \) by “wobbling” the table so that the first leg is touching the floor. Thus, \( S_1 \) is positive and \( S_2 \) is negative.
17. Yes, because any 3 points define a plane.
18. Yes, one can. Define lines on opposite legs of the 4-legged three-dimensional table. This means that the slope would be 0 somewhere in between and the table would be stabilized.
19. At 90°, the slopes \( S_1 \) and \( S_2 \) exchange their previous values. So, if \( S_1 \) was 0, it would become \(-k\) and if \( S_2 \) was \(-k\), it would become 0.
20. Since the values of \( S_1 \) and \( S_2 \) exchanged values, then \( S \) changed from \( k \) to \(-k\). It must have been 0 in between. Thus, the table can be stabilized within a 90° rotation.
21. It always is possible to stabilize a 4-legged three-dimensional table. This result is usually surprising.
Please keep in mind that “stabilize” can have two different interpretations. One interpretation is that all the legs of the table are on the floor at the same time so that it doesn’t take somebody’s foot to hold the table down or a napkin stuffed under a short leg. Another is that the tabletop is also horizontal so that nothing will slide off of it. Generally speaking, the first interpretation tends to apply to the three-dimensional table, and the second to the two-dimensional table.

We want to take a more careful look — you might even say “rigorous” look — at the mathematics underlying the simplest form of this modeling problem. Let us assume that the floor covers the interval \([0, 1]\) and that the height of the floor is given by a continuous function \(h(x)\). We assume that \(h(0) = h(1)\). Let the table have length \(1/2\). Does it follow that there must be an \(x \in [0, 1]\) such that \(h(x + 1/2) = h(x)\)? That would be a stable position of the table. It does follow, and the proof is given below.

**Proof:** Let \(g(x) = h(x + 1/2) - h(x)\), which is defined for \(x \in [0, 1/2]\). Either \(g(0) = 0\) or it doesn’t. If it \(g(0) = 0\), then \(x = 0\) is a value of \(x\) with the desired property. If \(g(0) \neq 0\), then we may assume without loss of generality that \(g(0) > 0\). Then we claim that \(g(1/2) < 0\). Why? Well, \(g(0) + g(1/2) = h(1/2) - h(0) + h(1) - h(1/2) = h(1) - h(0) = 0\), and so if \(g(0) > 0\), then \(g(1/2) < 0\). But \(g(x)\) is a continuous function because \(h(x)\) is continuous. Hence by the Intermediate Value Theorem, there is a value of \(x_0 \in (0, 1)\) such that \(g(x_0) = 0\). By definition of \(g\), \(h(x_0 + 1/2) = h(x_0)\).

A very similar argument will work for a table of length \(1/3\). We set \(g(x) = h(x + 1/3) - h(x)\). Then \(g(0) + g(1/3) + g(2/3) = 0\), and if \(g(0) > 0\), then at last one of \(g(1/3)\) and \(g(2/3)\) must be negative. Therefore \(g(0) = 0\) somewhere in \([0, 2/3]\). The same argument will work for a table of length \(1/n\), where \(n\) is an integer.

The result is false for a table of length \(\alpha\) if \(\alpha > 1/2\). For example, let \(h(x) = x\) in the interval \((0, 1 - \alpha)\), \(h(x) = (x - 1)\) from \(a\) to \(1\), and continuous in the middle.

**Question:** What happens if \(\alpha = 2/5\), or any rational number less than \(1/2\) and not of the form \(1/n\)? Does there have to be an \(x\) such that \(h(x + 2/5) = h(x)\)? No, there doesn’t! And there cannot be. For the proof of this see “A Stable One-Dimensional Table” in *Consortium*.
